How many megapixels does *your* radio interferometer have?

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CalIm, Cape Town South Africa, Dec 2012

Information in radio interferometry

- Image metrics
  - Dynamic range => e.g. 1000000:1 DR = 20 bits
  - Resolution => 1'' res = 2×20 bits
  - Number of pixels => 1 Megapixel ×8 bit = 1 Mbyte (trivial but somewhat arbitrary)
- Imaging algorithms
  - Maximum entropy method
  - Compressed sensing
- Interferometer networks
  - SKA network bandwidth comparable to global Internet trafic
  - Final data files (visibilities) has size on disk: order of Terabytes

Can one determine how much information are we getting from the sky into the interferometer?

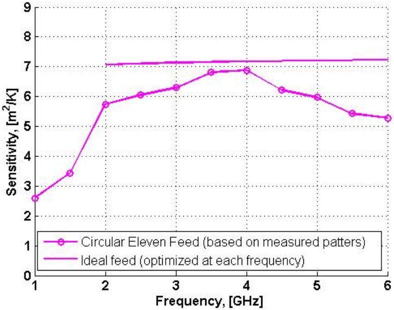
- Borrow ideas from telecoms (MIMO) and extending concepts from time domain to spatial
- View interferometer as spatial filter
- Compute how much information an interferometer can extract spatially from sky

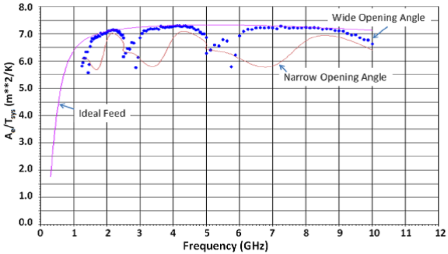
Definition

Interferometer channel capacity is the maximum amount of useful image information that can be extracted with an interferometer (measured in bits per spectral band and per observation)

The intuition is that an interferometer with a *large* imaging capacity is *better* than one with a *small* imaging capacity

Example: what constitutes a good filter?





Filter spectrum

The output $\{Y_i\}_1^n$ of a filter $\{a_i\}_1^n$ on an input $\{Y_i\}_1^n$ can be approximated in the time domain as the convolution

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_i \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} a_1 & \cdots & a_i & \cdots & a_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{-i} & \ddots & a_1 & \ddots & a_i \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{-n} & \cdots & a_{-i} & \cdots & a_1 \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_i \\ \vdots \\ X_n \end{bmatrix} \iff \mathbf{Y} = \mathbf{A}\mathbf{X}$$

This can be diagonalized by a Fourier transform (an unitary matrix) but, because for interferometers sampling may not be regular, do it as SVD

$$\mathbf{Y} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H\mathbf{X} \iff \tilde{\mathbf{Y}} = \boldsymbol{\Sigma}\tilde{\mathbf{X}}$$

where $\boldsymbol{\Sigma}$ is a diagonal matrix with diagonal entries $\{\sigma_i\}_1^n$ a.k.a the *singular values* and in this case is equivalent the square-root of the power spectrum of the filter.

Filter channel capacity

The channel capacity of the filter for a unit spectral denisty white noise signal input and additive white noise with spectral density N is the mutual information/entropy

$$C = \sum_{i=1}^{n} \log_2 \left(1 + \frac{\sigma_i^2}{N}\right)$$

in units of bits per n samples.

Suggestion

The more information transfered through the filter (channel capacity), the better the filter

Interferometer as a spatial frequency filter

Two sampling perspectives possible: voltage sampling or visibilities sampling.
Simple model of MEq: for $N = n(n-1)$ scalar visibilities and m point sources and equal gains G, then

$$\begin{bmatrix} \mathscr{V}_{12} \\ \vdots \\ \mathscr{V}_{pq} \\ \vdots \\ \mathscr{V}_{(n-1)n} \end{bmatrix} = G \begin{bmatrix} \mathrm{e}^{\mathrm{i}u_{12}l_1} & \dots & \mathrm{e}^{\mathrm{i}u_{12}l_s} & \dots & \mathrm{e}^{\mathrm{i}u_{12}l_m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathrm{e}^{\mathrm{i}u_{pq}l_1} & \dots & \mathrm{e}^{\mathrm{i}u_{pq}l_s} & \dots & \mathrm{e}^{\mathrm{i}u_{pq}l_m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathrm{e}^{\mathrm{i}u_{(n-1)n}l_1} & \dots & \mathrm{e}^{\mathrm{i}u_{(n-1)n}l_s} & \dots & \mathrm{e}^{\mathrm{i}u_{(n-1)n}l_m} \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ B_s \\ \vdots \\ B_m \end{bmatrix}$$

$$\mathscr{V} = G\mathsf{MB}$$

Interferometer Channel Capacity

The SVD of interferometer spatial frequency filter matrix $\mathbf{M}$

$$\mathbf{M} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H$$

gives the singular values $\mathrm{diag}(\boldsymbol{\Sigma})$. The singular values can be used to compute the interferometer channel capacity

$$C_{\mathrm{interf}} = \sum_{i=1}^{n(n-1)/2} \log_2\left(1 + \frac{G^2\sigma_i^2 B_i^2}{N^2}\right)$$

with unit sky temperature, and noise power N.
Taking the null hypothesis of a homogenuous sky at detection threshold ($GB/N = 1$), one obtains

$$C^0_{\mathrm{interf}} = \sum_{i=1}^{n(n-1)/2} \log_2\left(1 + \sigma_i^2\right)$$

which is a purely geometric quantity. We will refer to this a the *threshold spatial capacity* of the interferometer.

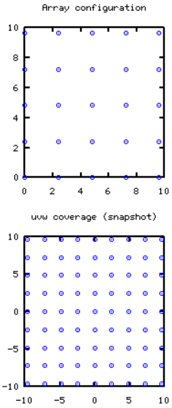
Gallery of Examples

I present some example element distributions along with their threshold spatial capacity. All examples have the same following assumptions

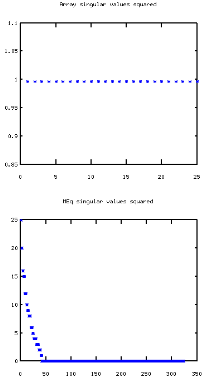
Parameter	Value
Number of elements	25
Maximum baseline	12λ
Field of view	polar cap 12° radius
Source Field	homogenous with flux at sensitivity threshold
Element pattern	Isotropic, unity gain

Uniform Rectangular Array (URA)

Distribution



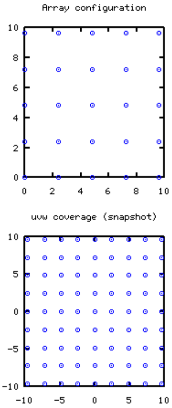
Singular value spectrum



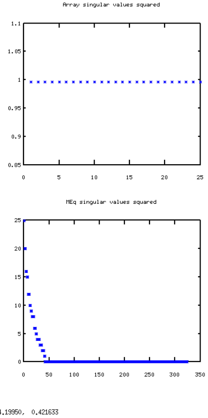
Array channel capacity 25 bit/snapshot MEq channel capacity 117 bits/snapshot

Uniform Rectangular Array (URA)

Distribution



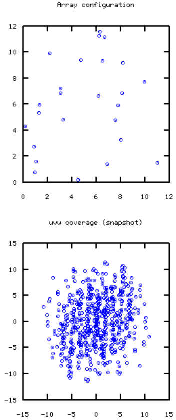
Singular value spectrum



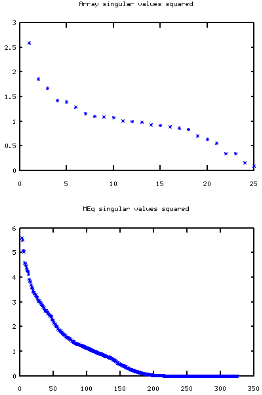
Array channel capacity 25 bit/snapshot MEq channel capacity 117 bits/snapshot

Poissonian

Distribution



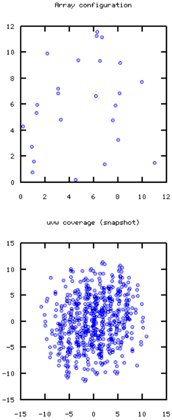
Singular value spectrum



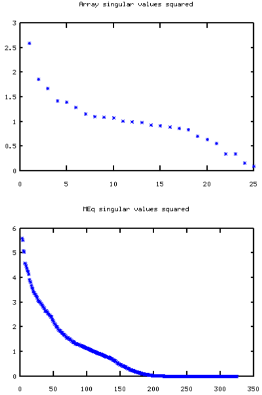
Array channel capacity 24 bit/snapshot MEq channel capacity 230 bits/snapshot

Poissonian

Distribution



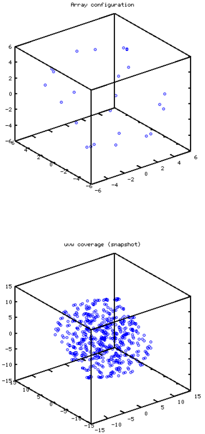
Singular value spectrum



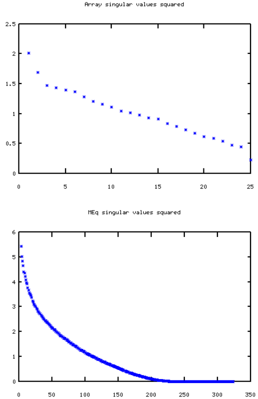
Array channel capacity 24 bit/snapshot MEq channel capacity 230 bits/snapshot

Poissonian on sphere

Distribution



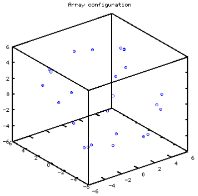
Singular value spectrum

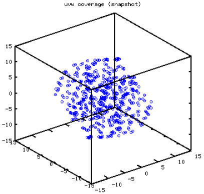


Array channel capacity 24 bit/snapshot MEq channel capacity 238 bits/snapshot

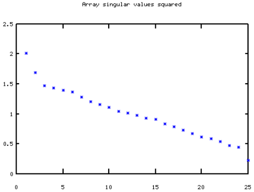
Poissonian on sphere

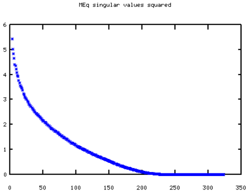
Distribution





Singular value spectrum





Array channel capacity 24 bit/snapshot MEq channel capacity 238 bits/snapshot

Additional uses of array capacity

Can be used to:

- compute information leakage (i.e. capacity over whole hemisphere compared with FoV)
  - e.g. for dipole elements with Poisson dist. (P=25), interferometer capacity is 477 bits over hemisphere but 122 bits are within element FWHM FoV
- can give real estimates of interferometer network capacity requirements
  - given spectral (bandwidth) capacity, system noise, gain, and sky distribution, a total information capacity can be computed

Important things to note about array capacity

- No mention of what imaging algorithms achieve maximum array capacity
- No mention about dirty synthesized beam pattern (no sidelobe levels etc)

Conclusions

I have introduced spatial channel capacity for interferometers. It

- is imaging algorithm independent (e.g. not based on dirty beams)
- intuitive quantity
- can be useful in the design of interferometer array configurations
  - Combines the concepts of DR, resolution and pixel number